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The Admittance of Cylindrical Antennas Driven From a Coaxial Line

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The situation in which the internal conductor of a coaxial transmission line projects through a ground-plane without any change in diameter is analyzed with reference to the infinite antenna and the parallel-plate antenna. An experimentally meaningful value of the admittance which takes into account the geometry of the antenna-to-coaxial line junction is obtained. This is achieved by utilizing a mathematical model in which an annular aperture or "gap" is represented by a frill of of magnetic current. The field distribution in the aperture is assumed to be of the same form as the TEM mode alone in the coaxial line. It is shown that the effect on the admittance of neglecting the TM modes in the line at the aperture, is very small. In the case of the parallel-plate antenna, the theoretical results are shown to agree remarkably well with experiment. It is also shown how the theory may be applied to the problem of finite length cylindrical antennas.

1. Introduction

The outstanding problem of cylindrical-antenna theory today is to determine a realistic value for the antenna admittance which takes into account the geometry of the feed system. In the past, attention has been focused primarily on the evaluation of the current distribution on the antenna proper. The nature of the feed system has in general been ignored, and its effect obliterated by the use of an infinitesimal "gap." As is well known, this type of source leads to an infinite value for the input susceptance of the antenna when the admittance of a symmetrical antenna, say, is defined by the equation, $Y=I(0)/V$. Attempts to overcome this obviously non-physical result have often been directed towards subtracting the singularity in some meaningful way (e.g., Chen and Keller, 1962; Duncan, 1962; Rao, 1965). The result of some other investigations has been to "smooth out" the singularity (e.g., Duncan and Hinchey, 1960). Some authors have bypassed it by using a variational technique (e.g., Storer, 1950; Rao, 1962), and some have defined the admittance by the equation, $Y=I(d)/V$, where $d\neq 0$, so that Y is no longer singular (e.g., Infeld, 1947). These methods are unsatisfactory in that reasonable correlation of theory with experiment is either fortuitous, since the results are invariant with respect to the actual aperture size, empirical, or obtained by way of arbitrary approximations.

In order to develop a rational and practical theory it is first necessary to decide upon an antenna-feed configuration which gives rise to an analytically convenient mathematical model without sacrificing practical usefulness. It is also necessary to select the most convenient and most widely applicable mathematical method in order that a range of problems encountered in practice might be dealt with in a unified way. Finally, it is advantageous if experimental results are available for an antenna configuration for which the mathematical analysis is particularly simple in order to verify the theory. All these requirements may be met in the following way: First, we use the feed geometry of figure 1, the mathematical model of which is shown in figure 2. This model was introduced by Albert and Synge (1948). Second, we use the Fourier-transform method. Third, in order to verify the theory, we analyze the mathematically simple parallel-plate antenna.

With regard to the use of the frill model, it is necessary to specify the tangential electric field in the aperture or "gap". Thus we neglect all the TM modes in the coaxial line at the aperture and take the field distribution to be that of the remaining TEM mode. A plausible argument is advanced to show that this approximation should have only a small effect on the resultant value

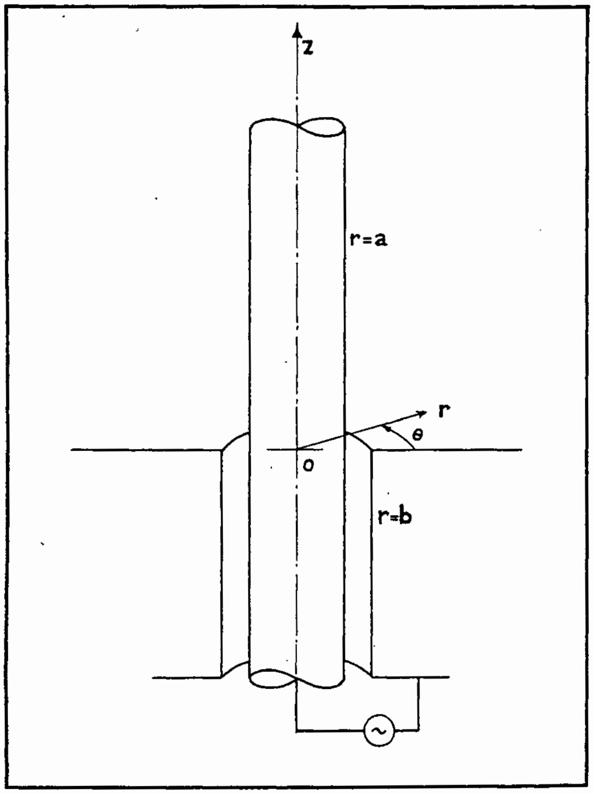

FIGURE 1. *Coaxial line to antenna geometry.*

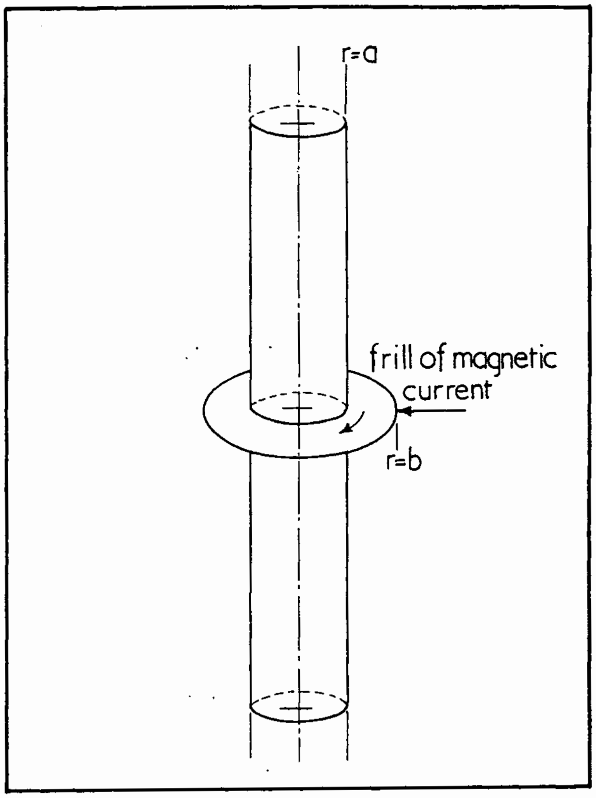

FIGURE 2. *Mathematical model of the antenna.*

for the admittance. This argument is reinforced by an analysis of the effect of individual TM modes on the admittance.

Concerning applications, the analysis of the physically unavailable infinite antenna is of fundamental significance when the Fourier-transform method is used, for the results therefrom may be applied in the solution of problems concerning both parallel-plate and finite-length antennas. Thus the first problem considered in this paper is that of the infinite antenna.

2. The Infinite Antenna with Coaxial Feed

The antenna under consideration may be taken to be an infinite, perfectly conducting rod of radius a mounted vertically above a highly conducting ground plane and fed coaxially from below, as shown in figure 1. The cylindrical coordinates (r, θ, z) relative to this antenna are also shown. Time variation is assumed to be harmonic and proportional to $e^{i\omega t}$.

Looking at the problem as a whole shows that there is an essential dichotomy in the type of information we want to extract. First, in order to evaluate the radiation characteristics of the system, we must determine the current distribution on the cylindrical part of the antenna as well as the total radiation effect of the transmission line. Second, in order to facilitate design of associated electronic equipment, we need to know the admittance the antenna presents to the generator. The first problem may be called the "exterior" problem and the latter, the "interior" problem.

This dichotomy suggests that although the two problems are uniquely related, a different approach may be warranted in each case. Thus we consider first the interior problem.

2.1. The Interior Problem

Let us effect the division between the interior and exterior problems by means of a fictitious surface across the aperture at $z=0$. Suppose, for the moment, that **E, H** are known on this surface. Then the interior problem is seen to be one of determining the proper combination of incident and reflected waves which will give rise to the given **E, H** on the terminating surface.

If the usual assumption of perfect conductivity of the coaxial line is made, and if it is also assumed that all the fields are axially symmetric and that the only nonzero components of the field are E_r, E_z, H_θ, then H_θ satisfies the wave equation,

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}+k^2\right\}H_\theta=0, \tag{1}$$

where

$$k^2=\omega^2\mu_0\epsilon_0. \tag{2}$$

Once the solution of (1) is known, the remaining field components may be determined from Maxwell's equations. Thus, when this is done, and the boundary conditions $E_z(a, z)=0$, $E_z(b, z)=0$ applied, we find that

$$E_r(r, z)=-\frac{k}{\omega\epsilon_0 r}\left\{A_0e^{ikz}-B_0e^{-ikz}\right\}-\frac{1}{i\omega\epsilon_0}\sum_{n=1}^{\infty}B_ne^{\gamma_n z}C_1(q_nr), \tag{3}$$

$$H_\theta(r, z)=\frac{1}{r}\left\{A_0e^{ikz}+B_0e^{-ikz}\right\}+\sum_{n=1}^{\infty}B_ne^{\gamma_n z}C_1(q_nr), \tag{4}$$

where

$$C_m(q_nr)=J_m(q_nr)Y_0(q_na)-Y_m(q_nr)J_0(q_na). \tag{5}$$

The q_n are roots of the equation

$$C_0(q_nb)=0, \tag{6}$$

J_n and Y_n are Bessel functions, and

$$\gamma_n^2=q_n^2-k^2. \tag{7}$$

At $z=0$, we have

$$E_r(r,0)=-\frac{Z_0}{r}(A_0-B_0)-\frac{1}{i\omega\epsilon_0}\sum_{n=1}^{\infty}\gamma_nB_nC_1(q_nr), \tag{8}$$

$$H_\theta(r, 0)=\frac{1}{r}(A_0+B_0)+\sum_{n=1}^{\infty}B_nC_1(q_nr). \tag{9}$$

If we integrate both sides of both these equations with respect to r from $r=a$, to $r=b$, it is easily verified that

$$\int_a^b E_r(r,0)dr=-Z_0(A_0-B_0)ln(b/a), \tag{10}$$

$$\int_a^b H_\theta(r, 0)dr = (A_0 + B_0)ln(b/a). \tag{11}$$

Therefore, since the admittance of the terminated line is given by

$$Y = \frac{2\pi}{Z_0 ln(b/a)} \left[\frac{B_0 - A_0}{B_0 + A_0}\right],$$

we find that

$$Y = \frac{2\pi}{Z_0 ln(b/a)} \frac{\int_a^b H_\theta(r, 0)dr}{\int_a^b E_r(r, 0)dr}. \tag{12}$$

This is a most important result for it shows that the admittance is related to the integrals of the tangential fields in the aperture.

Let us now assume that the contribution to the admittance from the TM modes in the coaxial line at the aperture is small. Thus the tangential electric field in the aperture may be taken to be

$$E_r(r, 0) = \frac{V}{r \ln(b/a)}, \tag{13}$$

where V is the applied voltage across the line.

A plausible argument in support of this approximation is as follows: First, as $r \rightarrow a$, the E_r field becomes proportional to $1/r$ provided ka is not too large. Second, as $r \rightarrow b$, it is evident that the E_r field in the aperture must deviate from the assumed distribution. Indeed there must be an edge singularity here as $r \rightarrow b$. However, it is clear from (10) that only the TEM mode in the electric field contributes to the integral of E_r in (12). Moreover, it is clear from (8) and (9) that H_θ is "smooth" compared with E_r because of the factor γ_n in (8) and its absence in (9). Because of this, and because Y is related to the integral of H_θ in the aperture so that oscillatory disturbances tend to integrate to zero, perturbations from the assumed distribution will give rise to smaller perturbations in the admittance. In other words, the main contribution to the result for the admittance is from the TEM mode in the coaxial line. The effect of the TM modes should be slight.

Thus, assuming this argument to be valid (further consideration of the point is deferred until section 4) and using (13), we find that the admittance is given by

$$Y = \frac{2\pi}{V \ln(b/a)} \int_a^b H_\theta(r, 0)dr. \tag{14}$$

2.2. The Exterior Problem

Subject to (13), the exterior problem is completely specified by the following boundary conditions:

$$E_z(a, z) = 0, \; z > 0,$$

$$E_r(r, 0) = \begin{cases} \dfrac{V}{r \ln(b/a)}, & a < r < b, \\ 0, \; r > b, \end{cases}$$

as well as the usual radiation conditions at infinity. It follows that the only nonzero components of the resultant field are E_r, E_z, H_θ.

Evidently we may have recourse to the method of images to simplify the problem. This gives rise to the situation shown in figure 2 where the infinite antenna is fed from a frill of magnetic current,

$$\mathbf{L}=-\mathbf{u}_z\times\{\mathbf{E}(r,\,0+0)-\mathbf{E}(r,\,0-0)\}\cdot \tag{15}$$

This gives

$$L_\theta=\begin{cases}-\dfrac{2V}{r\,ln\,(b/a)},\ a<r<b,\\ 0,\ r>b.\end{cases} \tag{16}$$

2.3. The Current on the Infinite Antenna

We are now in a position to determine the current distribution on this antenna. To achieve this we observe that the electromagnetic field is determined if the electric and magnetic vector potentials **A** and **M** are known. Thus

$$\mathbf{E}=\frac{1}{i\omega\mu_0\epsilon_0}\{\nabla\nabla\cdot+k^2\}\mathbf{A}-\frac{1}{\epsilon_0}\nabla\times\mathbf{M}\cdot \tag{17}$$

Because of the symmetry of this problem, it is evident that the only nonzero components of **A** and **M** are A_z and M_θ. Consequently

$$E_z=\frac{1}{i\omega\mu_0\epsilon_0}\left(\frac{\partial^2}{\partial z^2}+k^2\right)A_z-\frac{1}{\epsilon_0 r}\frac{\partial}{\partial r}(rM_\theta), \tag{18}$$

and A_z and M_θ are to be determined from

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}+k^2\right\}A_z=-\mu_0K_z(z)\delta(r-a), \tag{19}$$

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}+k^2\right\}M_\theta=-\epsilon_0L_\theta(r)\delta(z), \tag{20}$$

where K_z is the surface current density on the cylinder. Let us now take the Fourier transform of (18), (19), and (20). This transform, where it exists, is defined by the relation,

$$\mathscr{E}(r,\,\alpha)=\int_{-\infty}^{\infty}E(r,\,z)e^{-i\alpha z}dz. \tag{21}$$

Thus we find, for instance, that

$$\mathscr{A}_z(r,\,\alpha)=\frac{\mu_0\mathscr{I}}{2\pi}(\alpha)I_0(qa)K_0(qa). \tag{22}$$

Here $\mathscr{I}(\alpha)$ is the transform of the total current at any cross section $z=$constant. Also I_0 and K_0 are modified Bessel functions, $q=(\alpha^2-k^2)^{1/2}$, and the branch of the square root which is positive for large positive α is taken. Moreover, the wavenumber k is assumed to have a small negative imaginary part so that the Fourier transforms exist in the ordinary sense.

Solution of the transform of (20) is facilitated by solving for the case of the delta function right-hand side with

$$L_\theta(r)=\delta(r-r'). \tag{23}$$

This gives rise to $\mathscr{M}_\theta^*(r, r', \alpha)$, say, and then

$$\mathscr{M}_\theta(r, \alpha)=\int_a^b \mathscr{M}_\theta^*(r, r', \alpha)L_\theta(r')dr'. \tag{24}$$

Thus carrying out the indicated calculations and applying the boundary condition that $E_z(a, z)=0$ in (18), we obtain the result,

$$\mathscr{I}(\alpha)=\frac{4\pi ikV}{Z_0\, ln\,(b/a)q^2}\left[1-\frac{K_0(qb)}{K_0(qa)}\right]. \tag{25}$$

Therefore the current on the infinite antenna is given by

$$I(z)=\frac{2ikV}{Z_0\, ln\,(b/a)}\int_{\Gamma_1}\frac{e^{i\alpha z}}{q^2}\left[1-\frac{K_0(qb)}{K_0(qa)}\right]d\alpha, \tag{26}$$

where Γ_1 is the contour shown in figure 3. Note that this result may be inaccurate for small z because, although the effect of the TM modes on the admittance appears to be small, this is not necessarily the case with regard to the current.

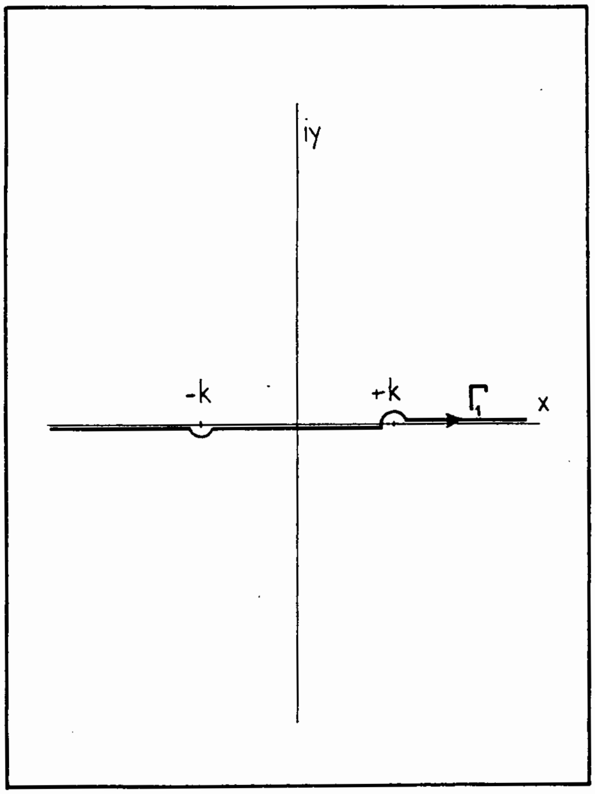

FIGURE 3. *Contour of integration for Fourier inverse.*

2.4. Numerical Evaluation of the Current

Numerical evaluation of $I(z)$ from (25) is facilitated by changing the contour of integration to Γ_2 as shown in figure 4, and making the change of variable, $\alpha=-ik\sqrt{v^2-1}$. The result is

$$I(z)=-\frac{4ikV}{Z_0\, ln\,(b/a)}\int_0^{\infty}\frac{e^{-kz\sqrt{v^2-1}}}{v\sqrt{v^2-1}}\times\left[\frac{J_0(vkb)Y_0(vka)-J_0(vka)Y_0(vkb)}{J_0^2(vka)+Y_0^2(vka)}\right]dv. \tag{27}$$

We have carried out the above integration using a 1620 digital computer. The result for $V=1/2$ V, $ka=0.02$, and $b/a=2.75$ is shown in figure 5. Note that we have plotted the amplitude term, $\psi(kz)$, in the equation

$$I(kz)=\psi(kz)e^{-ikz},\ z>0. \tag{28}$$

The corresponding amplitude term in the case of the infinitesimal gap is also shown in this figure. Note that the imaginary part of $\psi(kz)$ for the frill source is finite at $z=0$, whereas that for the infinitesimal gap is logarithmically singular.

2.5. Calculation of the Admittance

From (14) we see that the quantity which must be determined is $\int_a^b H_\theta(r,\ 0)dr$. To do this we observe that the magnetic field may be determined from

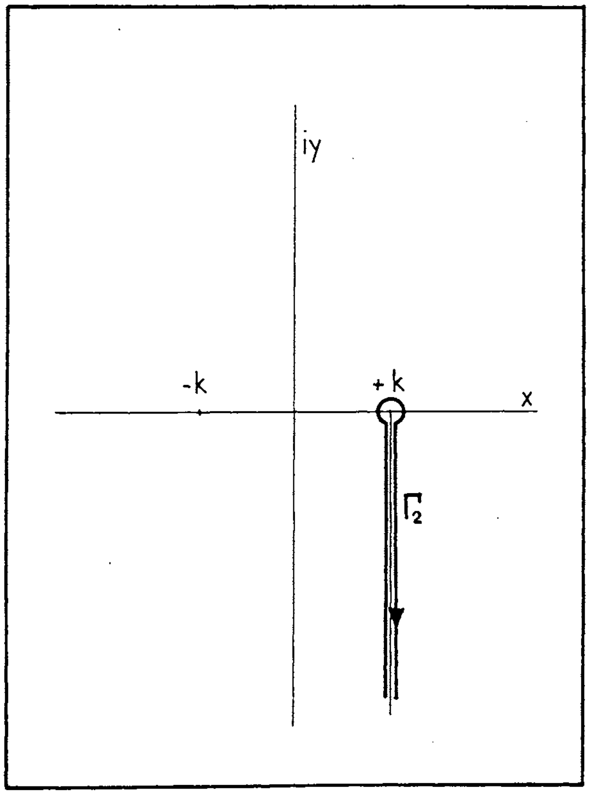

FIGURE 4. *Modified contour of integration.*

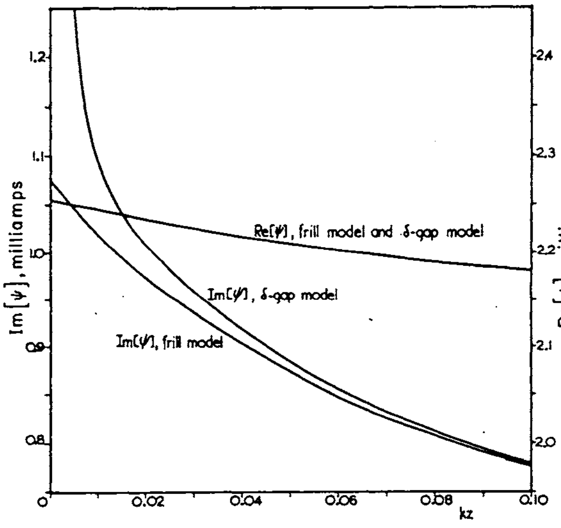

FIGURE 5. *Current amplitude on infinite antennas,* ka=*0.0664,* b/a=*2.25.*

$$\mathbf{H}=\frac{1}{i\omega\mu_0\epsilon_0}\left\{\nabla\nabla\cdot+k^2\right\}\mathbf{M}+\frac{1}{\mu_0}\nabla\times\mathbf{A}. \tag{29}$$

For this particular antenna we find that

$$H_\theta=-i\omega M_\theta-\frac{1}{\mu_0}\frac{\partial A_z}{\partial r}. \tag{30}$$

Therefore in terms of the Fourier transforms of the above quantities,

$$\int_a^b \mathscr{H}_\theta(r,\alpha)dr=-i\omega\int_a^b \mathscr{M}_\theta(r,\alpha)dr-\frac{1}{\mu_0}\{\mathscr{A}_z(b,z)-\mathscr{A}_z(a,z)\}\cdot \tag{31}$$

Using (24), it is easily verified that

$$\int_a^b \mathscr{H}_\theta(r,\alpha)dr=\frac{2ikV}{Z_0 ln(b/a)q^2}\left[ln(b/a)+\frac{K_0(qb)}{K_0(qa)}\{I_0(qa)K_0(qb)-I_0(qb)K_0(qa)\}\right]\cdot \tag{32}$$

Thus the admittance of the infinite antenna with frill source is given by Y_f, where

$$Y_f=\frac{2ik}{Z_0\{ln(b/a)\}^2}\int_{\Gamma_1}\left[ln(b/a)+\frac{K_0(qb)}{K_0(qa)}\{I_0(qa)K_0(qb)-I_0(qb)K_0(qa)\}\right]\frac{d\alpha}{q^2}. \tag{33}$$

Finally, in a similar manner to the development of (27) for the current, we find that a convenient representation for Y_f is

$$Y_f=\frac{2ik}{Z_0\{ln(b/a)\}^2}\int_0^\infty\frac{\{J_0(vka)Y_0(vkb)-J_0(vkb)Y_0(vka)\}^2}{v\sqrt{v^2-1}\,\{J_0^2(vka)+Y_0^2(vka)\}}dv. \tag{34}$$

3. The Parallel-Plate Antenna

Once the Fourier transform of the current on the infinite antenna has been obtained, it is a very simple matter to determine the current on a parallel-plate antenna of the configuration shown in figure 6. This may be done by constructing a periodic structure of frills of magnetic current exciting an infinite perfectly conducting rod. If these frills are located at $z=\pm 2nh$, $n=0, 1, 2 \ldots$, then on the planes $z=\pm(2n-1)h$, we get $E_r=0$. The parallel-plate antenna may be identified with one cell of this structure, namely, the region $0<z<h$, because the boundary conditions which characterize the antenna are satisfied.

Clearly, the current on the periodic structure is given by

$$I(z)=I_0+2\sum_{m=1}^{\infty} I_m\cos(m\pi z/h). \tag{35}$$

But the I_m are known to be periodic samples of $\mathscr{I}(\alpha)$, as given by (25), according to the equation,

$$I_m=\frac{1}{2h}\mathscr{I}(m\pi/h). \tag{36}$$

Thus the current on the parallel-plate antenna is

$$I(z)=\frac{2\pi iV}{Z_0kh\,ln\,(b/a)}\left[\left\{\frac{H_0(kb)}{H_0(ka)}-1\right\}+2\sum_{m=1}^{\infty}\frac{\cos\,(m\pi z/h)}{q_m^2}\left\{1-\frac{K_0(q_mkb)}{K_0(q_mka)}\right\}\right], \tag{37}$$

where

$$q_m=\sqrt{\left(\frac{m\pi}{kh}\right)^2-1}\cdot \tag{38}$$

The admittance can be determined from (14) and (32). The result is

$$Y=-\frac{2\pi i}{Z_0kh\,\{ln(b/a)\}^2}\left[ln(b/a)+\frac{\pi}{2}\,\frac{H_0(kb)}{H_0(ka)}\left\{J_0(kb)Y_0(ka)-J_0(ka)Y_0(kb)\right\}\right.$$
$$\left.-2\sum_{m=1}^{\infty}\frac{1}{q_m^2}\left\{ln(b/a)+\frac{K_0(q_mkb)}{K_0(q_mka)}\left[I_0(q_mka)K_0(q_mkb)-I_0(q_mkb)K_0(q_mka)\right]\right\}\right]\cdot \tag{39}$$

3.1. Numerical Results

Equations (37) and (39) were evaluated using a 1620 digital computer. Both series were truncated at $m=20$. Values of the current and admittance were calculated for $ka=0.0664$, $b/a=2.25$, in order that a comparison could be made with some experimental and theoretical results obtained by Rao (1962, 1965).

The current distribution for $kh=\pi/2$ is shown in figure 7. Clearly, our theory is satisfactory except perhaps near $z=0$, where the effect of neglecting the TM modes seems to be evident.

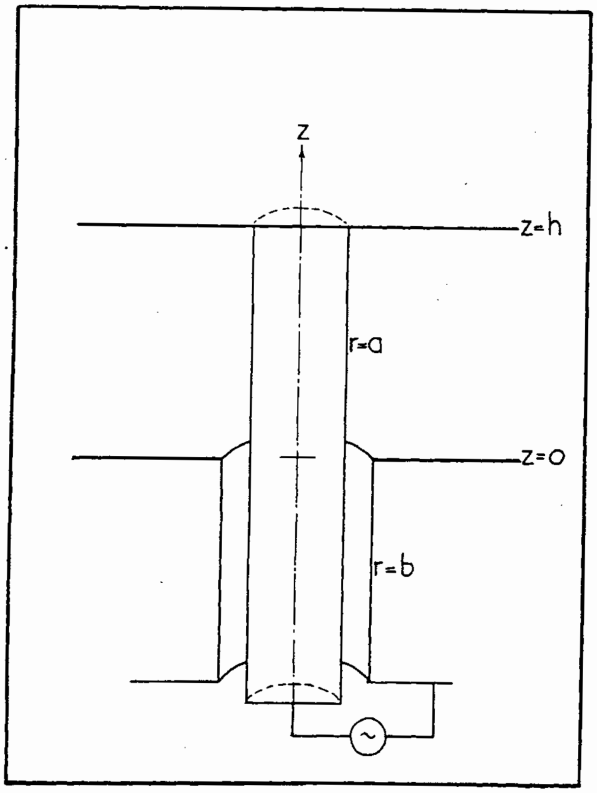

FIGURE 6. *Parallel-plate configuration.*

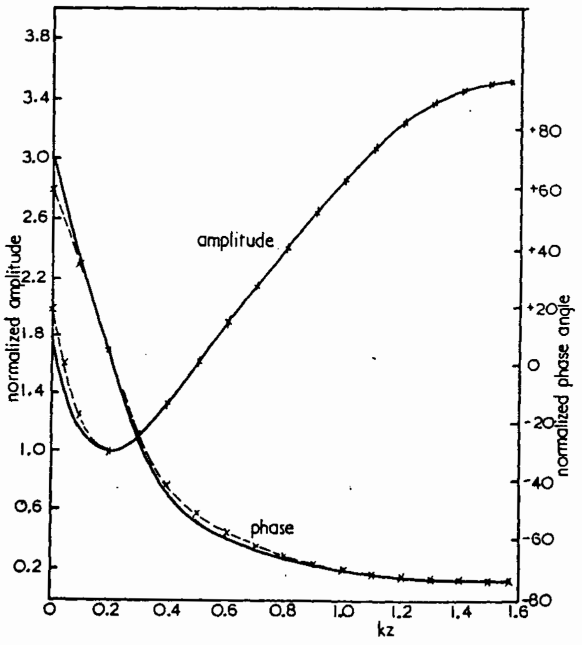

FIGURE 7. *Current on parallel-plate antenna.*
Experimental values (Rama Rao) ————
Theoretical values (Otto) — — × — —

In table (1), values of both the conductance, G_0, and the susceptance, B_0, obtained by the use of (39), are shown for a range of values of kh. Also shown in this table for the purpose of comparison, are the corresponding experimental values for the susceptance, B_e (Rao, 1965) and also some theoretical values. Of these, B_s denotes the values due to Rao (1965) when the singularity due to a delta-function source is subtracted. The term B_v denotes the values obtained when the singularity due to the same type of source is avoided by using a variational method (Rao, 1962, 1965).

TABLE 1. *Admittance in mmhos of the parallel-plate antenna,* ($Y=G+iB$).
$ka=0.0664$, $b/a=2.25$.

kh	G_0	B_0	B_e	B_s	B_v
0.7854	3.189	−2.885	−1.10	−5.49	−1.76
1.0000	2.504	−1.032	+0.20	−3.83	....
1.2690	1.932	+0.851	+1.60	−2.25	+1.45
1.5708	1.594	+2.318	+3.10	−0.99	+2.77
1.8450	1.357	+3.751	+4.50	+0.21	+4.04
2.0000	1.252	+4.628	+5.30	+1.06	+4.85
2.3562	1.063	+7.206	+7.60	+3.34	+7.23
3.5000	5.602	−8.371	−7.40	−12.41	....
3.9270	3.360	−2.086	−0.40	−6.23	....
4.3000	2.630	+0.232	+1.10	−3.98	....
4.6200	2.255	+1.717	+2.80	−2.55	....
4.7124	2.170	+2.122	+3.20	−2.17	....
4.9000	2.020	+2.954	+4.15	−1.37	....
5.1000	1.886	+3.908	+5.40	+0.47	....
5.4978	1.673	+6.428	+5.55	+2.00	....

The improvement resulting from the use of our theory is evident from a comparison of these results. It is to be noted that Rao's results, denoted by B_v, are the same whatever value of b/a is taken. Thus the apparent improvement in his results over ours in this case is fortuitous.

4. The Effect of the TM Modes on the Admittance

In order to estimate the effect of the TM modes on the admittance, consider the distribution,

$$E_{r_m}(r,0)=\frac{C_1(q_m r)}{aC_1(q_m a)\ln(b/a)}, \tag{40}$$

in which it is evident that

$$E_{r_m}(a,0)=E_{r_0}(a,0),$$

where

$$E_{r_0}(r,0)=\frac{1}{r\ln(b/a)},$$

so that the TM modes are roughly normalized with respect to the TEM mode. With this TM-mode excitation, the magnetic current on the frill is

$$L_{\theta_m}(r)=-2E_{r_m}(r,0). \tag{41}$$

Thus the method of sections 2.3 and 2.4 may be used to obtain the result,

$$\int_a^b \mathscr{H}_{\theta_m}(r,\alpha)dr=\frac{2ikb}{Z_0 a\ln(b/a)}\frac{C_1(q_m b)K_0(qb)}{C_1(q_m a)K_0(qa)}\frac{1}{q_m^2+q^2}\times\{I_0(qa)K_0(qb)-I_0(qb)K_0(qa)\}\cdot \tag{42}$$

Now let the distribution in the aperture be

$$E_r(r,\,0)=E_{r_0}(r,\,0)+E_{r_m}(r,\,0). \tag{43}$$

With this distribution, we can write the result for the admittance in the form,

$$Y=Y_0+Y_m, \tag{44}$$

where Y_0 is the admittance as calculated using the simple theory and may be identified with the first term on the right of (43). The value Y_m is the contribution to the admittance from $E_{r_m}(r,\,0)$. Thus, using the technique of section 3, the admittance of the parallel-plate antenna may be determined using (42). The result for Y_m is,

$$Y_m=\frac{2\pi i(b/a)}{Z_0kh\{ln\,(b/a)\}^2}\frac{C_1(p_mkb)}{C_1(p_mka)}\left[\frac{\pi}{2}\frac{1}{p_m^2-1}\frac{H_0^{(2)}(kb)}{H_0^{(2)}(ka)}\times\{J_0(kb)Y_0(ka)-J_0(ka)Y_0(kb)\}\right.$$

$$\left.+2\sum_{n=1}^{\infty}\frac{1}{p_m^2+q_n^2}\frac{K_0(q_nkb)}{K_0(q_nka)}\{I_0(q_nka)K_0(q_nkb)-I_0(q_nkb)K_0(q_nka)\}\right], \tag{45}$$

where $p_m=kq_m$ and the q_n are defined by (6).

The term Y_m was evaluated for $ka=0.0664$, $b/a=2.25$, and $m=1$, 2, and 3, for a few values of kh. The results are shown in table (2) along with Y_0 for comparison. Also shown is a series of values of G_0, B_0, G_1, B_1, for some of the values of kh given in table (1). Clearly, the contribution from the Y_m is small, thus verifying the argument put forward in section 2.1. Apparently this result is due to the factor $1/(p_m^2+q_n^2)$ in (45), because $p_m\approx m\pi/\{ka(b/a-1)\}$. Thus for small ka, Y_m will be small. Also note that the purturbation to the aperture distribution implied in (43) is quite violent. Physically, one would expect the actual deviation to be much smaller. Thus the resultant effect of the TM modes is probably much smaller than that indicated by table (2). This raises the question of the origin of the significant constant difference of about 1 mmho between B_0 and B_e in table (1). The point appears to be worthy of further consideration.

TABLE 2. *Effect of TM modes on the admittance of the parallel-plate antenna*, (Y = G + iB).

All admittance values in mmhos, $ka=0.0664$, $b/a=2.25$.

kh	G_0	B_0	m	G_m	B_m
1.0000	2.504	−1.032	1	.00265	.1377
			2	−.00066	−.0398
			3	.00029	.0185
2.0000	1.252	+4.628	1	.00132	.1215
			2	−.00033	−.0327
			3	.00015	.0148
5.0000	1.950	+3.418	1	.00146	.8333
			2	−.00036	−.0211
			3	.00016	.0094
0.7854	3.189	−2.885	1	.00373	.1410
1.5708	1.594	+2.318	1	.00169	.1285
3.5000	5.602	−8.371	1	.00176	.1001
3.9270	3.360	−2.086	1	.00171	.0946
4.3000	2.630	+0.232	1	.00163	.0904
4.6200	2.255	+1.717	1	.00155	.0870
4.9000	2.020	+2.954	1	.00148	.0843

5. The Admittance of a Finite Antenna

Einarsson's (1966) method may be used to show that the current on the cylindrical surface of a finite antenna of length h, driven by a frill source, is given by

$$I(z)=I_f(z)-\int_0^\infty e(z')\{I_e(z-h-z')+I_e(z+h+z')\}dz', \tag{46}$$

where $I_f(z)$ is the current on an infinite antenna with frill source, $I_e(z)$ is the current on a similar antenna with delta-function source, and $e(z)$ is the z component of the electric field on the cylinders $r=a, z>h$.

An approximate value for the admittance may be found if we observe that the right of (46) consists of an outgoing current wave and its "reflections" from the ends of the antenna. The H_θ field due to these reflections is approximately proportional to $1/r$ over the aperture provided the size of the aperture is small compared with the length of the antenna. Thus the admittance of the finite antenna is

$$Y=Y_f-2\int_0^\infty e(z')I_e(h+z')dz', \tag{47}$$

where Y_f is the admittance of the infinite antenna with frill source as given by (34).

6. Conclusion

We have shown that the use of a frill model for an antenna driven from a coaxial line, along with the approximation that the tangential electric field in the aperture is of the same form as the TEM mode in the line, leads to a theory which is easy to apply and gives results which correlate very well with experiment. In the case of the determination of the admittance, the extent of the above approximation has been assessed by evaluating the contribution to the admittance of individual TM modes in the aperture. This contribution was shown to be very small for small ka. Thus the theory has the distinct advantage over those mentioned in the introduction, that it gives a rational and accurate way of taking into account the dimensions of the coaxial aperture.

7. References

Albert, G. E., and J. L. Synge (1948), The general problem of antenna radiation and the fundamental integral equation, with application to an antenna of revolution, Pt. I, Quart. Appl. Math. **6,** No. 1, 117–131.

Chen, Y. M., and J. B. Keller (1962), Current on and input impedance of a cylindrical antenna, J. Res. NBS **66D** (Radio Prop.), No. 1, 15–21.

Duncan, R. H. (1962), Theory of the infinite antenna including the feedpoint singularity in antenna current, J. Res. NBS **66D** (Radio Prop.), No. 2, 181–187.

Duncan, R. H., and F. A. Hinchey (1960), Cylindrical antenna theory, J. Res. NBS **64D** (Radio Prop.), No. 5, 569–584.

Einarsson, O. (1966), A comparison between tube-shaped and solid cylinder antennas, IEEE Trans. Ant. Prop. **AP-14,** No. 1, 31–37.

Infeld, L. (1947), The influence of the width of the gap upon the theory of antennas, Quart. Appl. Math. **5,** No. 2, 113–132.

Rao, B. R. (1962), Theoretical and experimental studies of antennas and arrays in a parallel-plate region, Harvard Univ., Cruft Lab. Scientific Rept. 15 (series 2).

Rao, B. R. (1965), Current distribution and impedance of an antenna in a parallel-plate region, Proc. IEE, London **112,** No. 2, 259–268.

Storer, J. E. (1950), Variational solution to the problem of the symmetrical cylindrical antenna, Harvard Univ., Cruft Lab. Tech. Rept. 101.

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